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Stem analysis of 20 *Abies balsamea* (L.) Mill., 68 *Picea mariana* (Mill.) B.S.P., 19 *Picea glauca* (Moench) Voss, 31 *Populus tremuloides* Michx., and 37 *Betula papyrifera* Marsh. revealed form variation between species. A volume equation based on the paracone (a geometrical solid midway between a paraboloid and a cone) estimated individual tree volume within 10% of the true volume (at the 95% confidence level) for all species. The input variables required were total height and diameter at a relative height of 0.2 for *Betula papyrifera* and 0.3 for the other four species. If breast-height diameter was used, the effect of form variation on the accuracy of volume prediction was more pronounced. In this case, the geometrical equation modified for each species according to the average centre of gravity provided more consistently accurate volume estimates than either the paracone equation or Honer's transformed variable equation. For all species, the diameter measurement position was more critical than the version of the geometrical equation selected.

MacDonald, G. B., et R. R. Forslund. 1986. Application of a geometrical volume equation to species with different bole forms. Can. J. For. Res. **16**: 311–314.

Les analyses de tiges de 20 *Abies balsamea* (L.) Mill., 68 *Picea mariana* (Mill.) B.S.P., 19 *Picea glauca* (Moench) Voss, 31 *Populus tremuloides* Michx. et 37 *Betula papyrifera* Marsh. révèlent que la forme varie selon les espèces. Une équation de volume basée sur le paracône (un solide géométrique à mi-chemin entre la paraboloïde et le cône) a estimé le volume individuel de la tige à l'intérieur de 10% du vrai volume, pour un niveau de probabilité de 95% et ce pour toutes les essences. Les données requises étaient la hauteur totale et le diamètre à la hauteur relative de 0,2 pour le *Betula papyrifera* et de 0,3 pour les quatre autres espèces. Si nous utilisions le diamètre à hauteur de poitrin, l'effet de la variation de la forme sur la précision du volume prédit serait plus prononcé. Dans ce cas, l'équation géométrique modifiée pour chaque essence par rapport au centre de gravité moyen, procure une précision plus consistante pour l'estimation des volumes que l'équation du paracône ou l'équation à variable transformée de Honer. Pour toutes les essences, la position de la mesure du diamètre était plus critique que la version de l'équation géométrique choisie.

[Traduit par le journal]

Introduction

Forslund (1982) developed a volume equation for aspen (*Populus tremuloides* Michx.) based on the location of the centre of gravity of the bole. The general form of his equation is

$$[1] \quad V = [A/(A + 2)][1 - K]^{-2/A}[\pi D_K^2 H/40\,000]$$

where V is the gross total volume (cubic metres), K is the relative height of the diameter measurement, D_K is the outside-bark diameter (centimetres), and H is the total bole height (metres). The variable A is related to the centre of gravity ($\bar{K}$) of a geometrical solid as follows: $A = (2\bar{K})/(1 - 2\bar{K})$. Forslund used an A value of 1.5 in Eq. 1, derived from the average relative height of the centre of gravity of his samples ($\bar{K} = 0.3$). He termed the corresponding geometrical solid a "paracone," since its A value fell midway between that of a paraboloid and that of a cone. His paracone equation estimated the true volume of individual trees within 10% (at the 95% confidence level), using only the total height and the diameter at a relative height of 0.3. Similar results were obtained using diameter at breast height, provided breast height was between 20 and 60% of the total height. The accuracy and ease of application reported by Forslund (1982) make the equation a candidate for widespread use in research and management.

Gray (1956) defined form as the shape of the bole characterized by the exponent in its profile function and defined taper as the rate of narrowing in diameter in relation to increase in height of a given form. This distinction is adopted here to prevent confusion between geometrical form and form quotient or form class. Since the latter terms are simply ratios of two bole diameters (Husch *et al.* 1982), they quantify rate of taper rather than geometrical form.

The objectives of this study were (*i*) to examine the geometrical form of five species, balsam fir (*Abies balsamea* (L.) Mill.), black spruce (*Picea mariana* (Mill.) B.S.P.), white spruce (*Picea glauca* (Moench) Voss), white birch (*Betula papyrifera* Marsh.), and aspen larger than those sampled by Forslund (1982) and (*ii*) to test the accuracy of Eq. 1 for each of the above species, using a fixed A value of 1.5 (the paracone model) and using a variable A value calculated from the average centre of gravity of each species.

Methods

The 175 sample trees were obtained from three upland mixedwood stands in sections B.8 and B.11 of the Boreal Forest Region (Rowe 1972). The soils in all stands were fresh loams (Anonymous 1982). To reduce the influence of crown class on form, only dominant and codominant trees were selected. Their size data are presented in Table 1.

Each tree was felled, measured for total height, and sectioned for sample disks at breast height (1.3 m) and at 1-m intervals from ground level. All disks were kept frozen until measured to minimize radial shrinkage. The average outside-bark diameter of each disk was determined as the average of the longest and shortest diameters. This diameter was measured to the nearest 0.01 mm and stored on diskette using a Holman Digimicrometer and Apple II computer.

The true gross total volume of each tree was determined by fitting a quadratic equation to each 1-m section using three diameter points, integrating it to obtain the section volume, and summing the section volumes. Grosenbaugh (1966) described this approach, which provides a very accurate estimate of the true volume.

Two approaches were used to compare the geometrical form of the

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TABLE 1. Descriptive statistics for sample trees

Species	Sample size	Diameter at breast height		Total height		Relative height of $\bar{K}^a$	
		Mean (cm)	CV[b] (%)	Mean (m)	CV (%)	Mean	CV (%)
Balsam fir	20	13.3	36.6	12.8	24.9	0.295	5.5
Black spruce	68	15.1	24.7	15.7	21.1	0.318	5.4
White spruce	19	16.3	38.1	13.8	21.9	0.308	4.0
Trembling aspen	31	25.8	25.7	24.5	13.1	0.312	4.6
White birch	37	10.0	32.4	12.1	17.8	0.286	6.5

[a]Centre of gravity.
[b]Coefficient of variation.

five species. First, bole profile graphs were constructed to illustrate the average form of each species in relation to a cone and a paraboloid equal in volume to the bole. The procedure for deriving these profiles is outlined in the Appendix. The second approach was based on the average relative location of the centre of gravity of each species. This was obtained from the true volume and the moment of force of each tree, as described by Johnson and Kiokemeister (1967). Comparisons were made between the calculated centres of gravity of the five species with reference to the centres of gravity of standard geometrical solids. Multiple *t*-tests were applied to determine which pairs of species displayed significant differences between their centres of gravity.

The accuracy of the geometrical equation with various diameter measurement positions was examined using the χ^2 test of Freese (1960). For this purpose, diameters were interpolated at 10% intervals of height using a quadratic fit of the original diameter data.

The accuracy of Honer's transformed variable equation (Honer 1967) was similarly tested on the sample trees. This equation has been cited as an example of the recent trend toward volume equations that avoid a separate measure of form, requiring only breast-height diameter and total height as input variables (Clutter *et al.* 1983). The coefficients for these equations are derived through regression techniques. Honer's equation may be expressed as follows:

$$[2] \quad V = D^2/[b_0 + (b_1/H)]$$

where V is the gross total volume, D is the outside-bark diameter at breast height, H is the total height, and b_0 and b_1 are regression coefficients that differ for each species. Honer's equation was included as an accepted standard to help assess the performance of the geometrical equation.

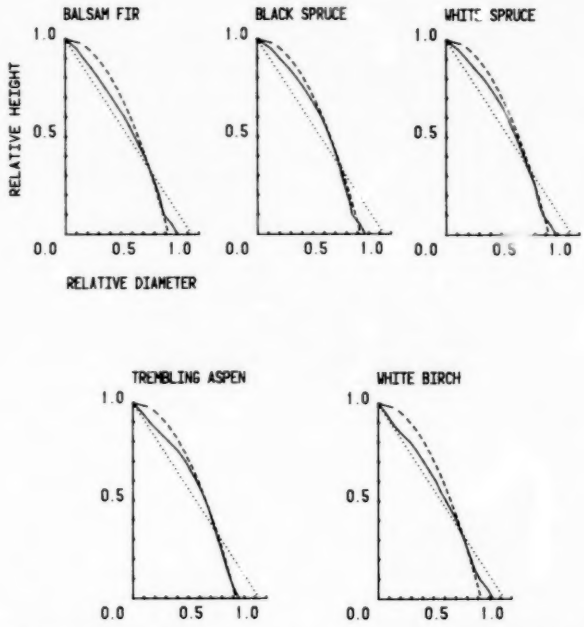

FIG. 1. The average bole profile (—) for each species sampled compared with a paraboloid (- - -) and cone (·····) equal in volume and height to the average bole.

Results and discussion

The profile of an annual growth layer is controlled by the amount of photosynthate and growth substances available at different relative bole heights. Farrar (1961) illustrated the importance of crown structure in this regard. Present bole form is the end result of the superposition of growth layers of varying profile throughout the life of the tree.

Since crown structure varies between species it is reasonable that bole form should also vary interspecifically. Form variation was revealed among the five species in this study (Fig. 1). Balsam fir closely resembled a paracone (the geometrical solid midway between a paraboloid and a cone). In comparison, black spruce, white spruce and aspen had larger average diameters between 0.40 and 0.70 of the height and smaller diameters between 0.10 and 0.30 of the height. These deviations caused the average form of black spruce, white spruce, and aspen to approach that of a paraboloid. White birch appeared slightly more conical than paraconical along most of its height.

These trends in form variation were confirmed by the average relative heights of the centres of gravity (Table 1). The value for balsam fir was close to that of a paracone (0.30). The centres of gravity of black spruce, white spruce, and aspen deviated toward the value for a paraboloid (0.33) and the centre of gravity of white birch deviated toward that of a cone (0.25).

Multiple *t*-tests revealed no significant differences at the 95% confidence level between the centre of gravity of aspen and that of black spruce or white spruce. This supports the previous results, which indicated that these three species all tended toward the paraboloidal form. The centres of gravity of all other species pairs differed significantly at the 95% confidence level.

The aspen in this study were more paraboloidal than those sampled by Forslund (1982). Since the former were also much taller, it appears that geometrical form may vary within a species according to size class. This would be expected considering the changes in live crown ratio and mechanical stress associated with changes in tree size. Variation in bole form between species and between dominance classes was reported by Reed and Byrne (1985).

In view of the interspecific form variation encountered, greater accuracy was expected from the geometrical equation (Eq. 1) when its A value was calculated from the centre of

TABLE 2. Freese's χ^2 test of estimated tree volumes

Species	% error ($\pm P\%$)[a]						
	Geometrical formula						Honer's formula, DBH
	$d_{0.1}$[b]	$d_{0.2}$	$d_{0.3}$	$d_{0.4}$	$d_{0.5}$	DBH[c]	
Balsam fir							
$A = 1.500$[d]	13.2	8.4	7.6	10.0	12.0	12.7	10.9
$A = 1.437$[e]	14.9	8.6	7.2	10.7	13.4	14.1	
Black spruce							
$A = 1.500$	23.8	15.4	9.5	11.7	23.7	23.8	24.0
$A = 1.750$	15.7	10.7	8.6	10.4	16.7	15.9	
White spruce							
$A = 1.500$	13.4	9.5	5.1	10.5	15.5	13.2	11.9
$A = 1.600$	10.2	7.7	5.0	10.0	13.3	10.3	
Trembling aspen							
$A = 1.500$	16.1	10.9	7.1	12.4	23.8	17.2	12.0
$A = 1.662$	12.1	8.2	7.0	11.3	18.5	12.8	
White birch							
$A = 1.500$	16.9	9.2	13.1	15.3	21.0	12.7	21.0
$A = 1.334$	17.3	10.9	12.7	16.8	23.0	14.6	

[a]Individual tree volumes can be estimated to within $P\%$ of the true value at the 95% confidence level.
[b]Diameter measured at a relative height of 0.1.
[c]Diameter measured at breast height.
[d]The value of A (Eq. 1) for a paracone.
[e]The value of A (Eq. 1) derived from the average centre of gravity for each species.

gravity of the species than when the fixed paracone A value of 1.5 was used. This expectation was partially supported by the results of the χ^2 tests as summarized in Table 2. When diameter was measured at breast height, it was more accurate to base the A value on the average centre of gravity in the case of black spruce, white spruce, and trembling aspen. With the other two species, slightly higher accuracy was achieved using a constant A value of 1.5. However, with balsam fir, white spruce, and aspen, Honer's equation provided volume estimates with comparable levels of accuracy.

Significant improvements in accuracy resulted for all species when single upper-stem diameters were used with the geometrical equation. With the exception of white birch, the best position for diameter measurement was the relative height of 0.3. Accuracy steadily decreased as the measurement position was moved in either direction from 0.3. For white birch the optimum relative height for measurement was 0.2. Since the profiles of all standard geometrical solids of equal volume converge between the relative heights of 0.25 and 0.35, form variation has the least effect on the accuracy of volume estimation when diameters are measured in or near this range.

For all species, the choice of diameter measurement position was more critical than the choice of A value in the geometrical equation. There was only a minor improvement in accuracy when the paracone A value of 1.5 was replaced with the value based on the average centre of gravity for the species, provided the diameter was measured at a relative height of 0.3. The minor improvement was insufficient to justify the cost of determining the exact centre of gravity for every species of interest. When the diameters were measured at the optimum relative heights, volume prediction errors using the simpler paracone equation ranged from 5.1% for white spruce to 9.5% for black spruce.

The results using relative diameter positions were more meaningful than those using breast-height diameters in assessing the applicability of the geometrical equation. Breast height fell at variable relative heights, incurring variable inaccuracies depending on the proximity to the 0.25–0.35 range of profile convergence. If breast-height measurements must be used, the geometrical equation using the A value based on the average centre of gravity is recommended for each species. This approach was associated with errors in the 10–16% range (Table 2). The use of breast-height diameter with the paracone equation ($A = 1.5$) or Honer's equation resulted in highly variable accuracy with errors as large as 24%.

If the 0.3 position is beyond reach from the ground, Finnish calipers should be considered. These are more accurate and rapid to use than optical instruments for obtaining upper-stem diameters (Schmid-Haas and Winzeler 1981). If upper-stem diameters cannot be obtained, a fixed measurement height of 2 m could be used rather than breast height to keep the diameters closer to the optimum relative height. The value of K in Eq. 1 would then become $2/H$, where H is the total tree height in metres.

Conclusions

The relatively small sample size in this study does not permit precise definitions of the characteristic geometrical forms of the five species examined. Factors such as site, size class, and stand structure influence form quotient (Gevorkiantz and Olsen 1955) and likely also influence geometrical form. Thus, a precisely defined, unique geometrical form for an entire species seems unattainable, regardless of sample size. For the samples in this study, balsam fir is close to a paracone, black spruce, white spruce, and trembling aspen are more paraboloidal, and white birch is more conical.

Despite form variations of the magnitude encountered here, the geometrical equation developed by Forslund (1982) can be used with a paracone A value of 1.5 to estimate individual tree

volumes within 10% (at the 95% confidence level) for the species considered. However, for this level of accuracy the single diameter measurement must be made at a relative height of 0.2 for white birch and 0.3 for the other four species. If the diameters cannot be obtained at the 0.2 and 0.3 positions, they should be measured above breast height and as close as possible to the optimum relative positions to minimize volume errors caused by form variation. If breast-height diameter must be used, the geometrical equation with a value of A based on the centre of gravity of the species is recommended over both Honer's equation and the geometrical equation with a fixed paracone A value of 1.5.

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Appendix

To eliminate the influence of tree size and taper and thus facilitate form comparisons, relative height was plotted against relative diameter. For scaling reference, the paracone was assigned a height of one unit and a basal diameter of one unit. Since volumes and relative total heights were constant, only the x-coordinate (relative diameter at a given relative height) varied between profiles. A constant volume of $(3/7)(\pi/4)$ cubic units was used (Eq. 1 with $A = 1.5$, $K = 0$, $D_K = 1$, and $H = 1$).

A generalized volume equation can be written as follows (Forslund 1982):

[A1] $$V = F_K B_K H$$

where V is the gross total volume, F_K is the normal form factor at a relative height K, B_K is the basal area at a relative height K, and H is the total height. From Eq. A1 it follows that $(\pi D_K^2)/4 = V/(F_K H)$, where D_K is the outside-bark diameter in the same units as H at a relative height K. Since V is $(3/7)(\pi/4)$ and H is 1, this simplifies to

[A2] $$D_K = \sqrt{(3/7)/F_K}$$

For the cone and paraboloid, $F_K = [A/(A + 2)][1 - K]^{-2/A}$, where A is 1.0 for a cone and 2.0 for a paraboloid (Forslund 1982). From Eq. A1 $F_K = V/(\pi D_K^2 H/40)$ for the actual bole profiles, where V is the true gross total volume (cubic decimetres). After applying Eq. A2 to each tree, average relative diameters for each species were calculated at 5% intervals of relative height to provide the coordinates for the profile graphs.